Identificar funções diferenciáveis

1 - Lineares: $f(x) = Ax$

$A_{m \times n}$ $\quad Df(x) = A$ //

$$f(x,y) = x = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

2 - Constantes: $f(x) = C \in \mathbb{R}^m$

$f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad Df(x) = 0$

3 - $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, dif. em a

$\Rightarrow$ $f+g$ dif em a

e $D(f+g)(a) = Df(a) + Dg(a)$

Exemplo: $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \underbrace{1}_{\text{constante}} + \underbrace{2x + 3y}_{\text{linear}}$$

—//—

Exemplo: $f(x,y) = xy$

—//—

4- Função composta

Exemplo: $f(x,y) = \sqrt{x^2+y^2}$

$$\begin{tikzcd}
(x,y) \arrow[r, mapsto, "s"] \arrow[rr, bend right, "f"'] & x^2+y^2 \arrow[r, mapsto, "r"] & \sqrt{x^2+y^2}
\end{tikzcd}$$

$f = r \circ s$

$$f(x,y) = \sqrt{x^2+y^2} = \|(x,y)\|$$

não é dif. em $(0,0)$.

<u>gráfico de f</u> :

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\quad f \quad} & \mathbb{R} \\ (x,y) & \longmapsto & \sqrt{x^2+y^2} \geq 0 \end{array}$$

$$\Big(\overbrace{x, y}^{\text{objecto}}, \overbrace{\sqrt{x^2+y^2}}^{\text{imagem}} \Big) = (x, y, f(x,y))$$

$$= (x, y, z) \ : \ z = f(x,y)$$

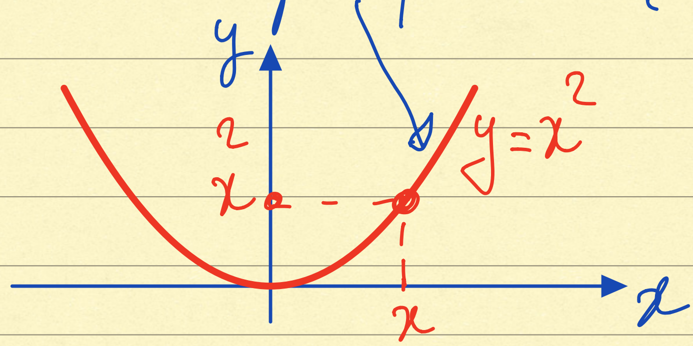
$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \big(u(x,y),\, v(x,y)\big)$$

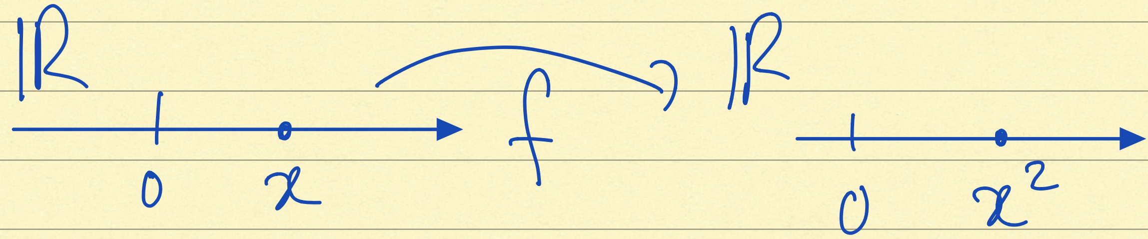
$$\big(x,\, y,\, u(x,y),\, v(x,y)\big)$$

Ex: $f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$ gráfico: $\{(x, x^2) : x \in \mathbb{R}\}$

$y = x^2$.





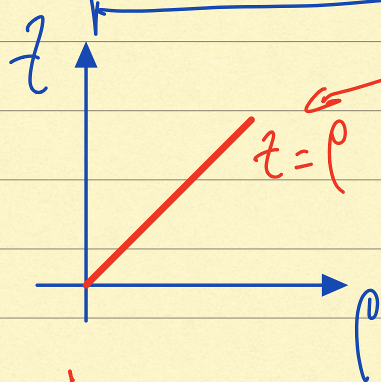
Ex: $f(x,y) = \sqrt{x^2+y^2}$

Gráfico: $(x, y, \sqrt{x^2+y^2})$

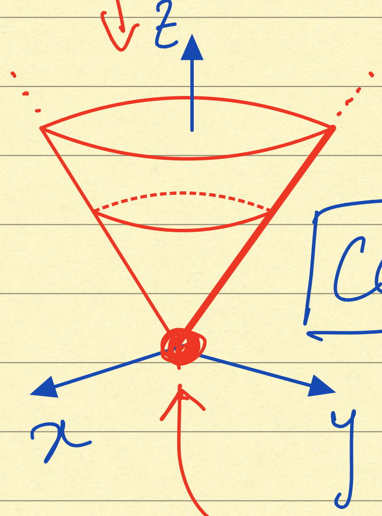
$$\boxed{z = \sqrt{x^2+y^2}} = \rho \quad \text{(rô)}$$

(dist. ao Oz)

(rodar)



Rodar



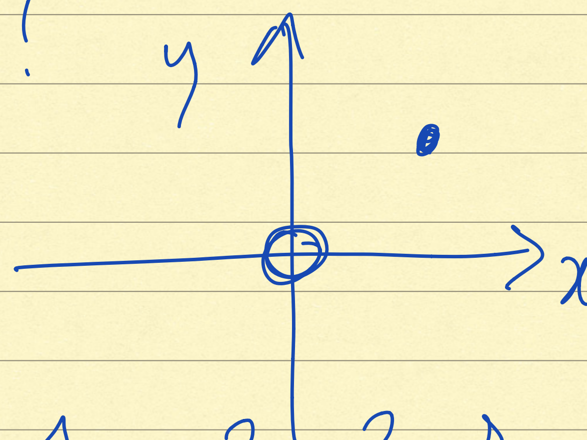
$\boxed{\text{CONE}}$

$\boxed{\text{Vértice}}$

f não é dif. em (0,0)!

f não é dif. em $(0,0)$.

Será dif. nos restantes pontos?



$$\begin{tikzcd}
(x,y) \arrow[r, mapsto, "s"] \arrow[rr, bend right, "f"'] \arrow[d] & \underbrace{x^2+y^2} \arrow[r, mapsto, "r"] & \sqrt{x^2+y^2} \\
\mathbb{R}^2 \arrow[r, "\text{dif}"] & \mathbb{R} \arrow[r, mapsto, "\text{dif.}"] & \mathbb{R} \\
 & r \arrow[r, mapsto] & \sqrt{r}
\end{tikzcd}$$

$$f = r \circ s$$

CDI-II $\qquad \boxed{r > 0} \qquad$ CDI-I

Caso geral: Teorema

Se

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"', "\text{dif. em } a"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^p \arrow[r, "f", "\text{dif. em } g(a)"'] & \mathbb{R}^m \end{tikzcd}$$

$$a \longmapsto g(a) \longmapsto f(g(a))$$

então

$f\circ g$ é dif. em a.

$$\boxed{\underbrace{D(f\circ g)(a)}_{m\times n} = \underbrace{Df(g(a))}_{m\times p}\,\underbrace{Dg(a)}_{p\times n}}$$

Exemplo: $f(x,y)=\sqrt{x^2+y^2}$

$$\begin{tikzcd} (x,y) \arrow[r, mapsto, "s"] \arrow[rr, bend right, "f=r\circ s"'] & x^2+y^2 \arrow[r, mapsto, "r"] & \sqrt{x^2+y^2} \end{tikzcd}$$

$$\underset{\neq (0,0)}{\overset{\mathbb{R}^2}{\boxed{(a,b)}}} \xmapsto{\;s\;} \overset{\mathbb{R}}{\boxed{a^2+b^2}} \xmapsto{\;r\;} \overset{\mathbb{R}}{\sqrt{a^2+b^2}}$$

$$f = r \circ s$$

$$\underset{1\times 2}{Df(a,b)} = \underset{1\times 1}{Dr(s(a,b))}\,\underset{1\times 2}{Ds(a,b)}$$

$$r(s) = \sqrt{s} = s^{\frac{1}{2}}$$

$$Dr(s) = \left[\frac{1}{2\sqrt{s}}\right]_{1\times 1} \longrightarrow \text{CDI-I}$$

$$s(x,y) = x^2+y^2$$

$$Ds(x,y) = \begin{bmatrix} 2x & 2y \end{bmatrix}_{1\times 2}$$

$$Df(a,b) = \left[\frac{1}{2\sqrt{a^2+b^2}}\right] \begin{bmatrix} 2a & 2b \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(a,b) = \frac{2a}{2\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\frac{\partial f}{\partial y}(a,b) = \frac{2b}{2\sqrt{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}$$

<u>Ex:</u> $f(x,y,z) = e^{x^2+y^2} + xz$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ dif. em $\mathbb{R}^3$

porque é a soma de duas dif.. Uma é composta, outra é um produto.

$$f(x,y,z) = \underbrace{e^{x^2+y^2}}_{g(x,y,z)} + \underbrace{xz}_{h(x,y,z)}$$

$$Df(x,y,z) = Dg(x,y,z) + Dh(x,y,z)$$

$$Dg(x,y,z) = ?$$

$$(x,y,z) \xmapsto{s} x^2+y^2 \xmapsto{e} e^{x^2+y^2}$$

$$Dg(x,y,z) = \left[e^{x^2+y^2}\right]\left[2x \quad 2y \quad 0\right]$$

$$= \left[2x e^{x^2+y^2} \quad 2y e^{x^2+y^2} \quad 0\right]$$

$$Dh(x,y,z) = \left[z \quad 0 \quad x\right]$$

$$Df(x,y,z) = \left[\, 2x e^{x^2+y^2} + z \quad 2y e^{x^2+y^2} \quad x \,\right]$$

Ex: $f(x,y) = g\big(u(x,y), v(x,y)\big)$

$u, v, g : \mathbb{R}^2 \to \mathbb{R}$, dif.

$$(x,y) \longmapsto \big(u(x,y), v(x,y)\big) \longrightarrow g\big(u(x,y), v(x,y)\big)$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$